Computers and Chemical Engineering 106 (2017) 572–581

Contents lists available at ScienceDirect

Computers and Chemical Engineering

journal homepage: www.elsevier.com/locate/compchemeng

# On anti-aliasing filtering and over-sampling scheme in system identification☆

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ARTICLE INFO

*Article history:*
Received 1 January 2017
Received in revised form 3 June 2017
Accepted 28 July 2017
Available online 12 August 2017

*Keywords:*
System identification
Sampling frequency
Anti-aliasing filtering
Variance
Over-sampling scheme

ABSTRACT

In this work, two fundamental issues in system identification, sampling frequency and anti-aliasing filtering are revisited, questions like, "Can higher sampling frequency help increase model accuracy?", and, "Can we do better than anti-aliasing filtering?" will be answered. First it is shown that, the traditional anti-aliasing filtering procedure does not give a consistent estimate of the desired model. Then an identification method is proposed based on the over-sampling scheme where a high frequency model is first identified and then converted to the (lower) control sampling frequency. It is shown that when the output noise contains energy beyond the bandwidth of the plant, the proposed method performs anti-aliasing in both open-loop and closed-loop identification; and in closed-loop identification, extra excitation is achieved from the high frequency noise. Simulation studies are used to illustrate the findings. The result has important implications in control applications.



## 1. Introduction

The basic requirement for system identification, especially in large-scale control applications, e.g. chemical process industries, is to obtain good quality models at a low cost. Noise aliasing is an issue that an identification experiment may have and will reduce the model quality. It happens when sampling the continuous-time process output that contains high frequency noise outside the frequency band of the plant (1/2 of the sampling frequency), according to the well-known Nyquist-Shannon sampling theorem. The noise aliasing involved in the output will then decrease the signal-to-noise ratio. Therefore, anti-aliasing measure is needed to avoid this noise aliasing.

The generally known way to avoid the noise aliasing is the anti-aliasing filtering procedure, see Ljung (1999). In this procedure, first the input and output are sampled at a high enough frequency to ensure the Nyquist sampling condition (the output noise contains no energy outside 1/2 of this sampling frequency) and give no chance to the noise aliasing; then the sampled signals are low-pass filtered to remove the frequency content beyond the frequency band of the plant; finally the filtered data are down-sampled to the needed sampling frequency for model estimation. Beside the anti-aliasing filtering procedure, the anti-aliasing problem is seldom discussed in the system identification literature. And, the anti-aliasing filtering procedure is lack of analysis, so we question that if this procedure is feasible in system identification.

On the other hand, to solve the anti-aliasing problem, we are inspired by the over-sampling scheme proposed in Sun et al. (1997), see also Sun and Sano (2009). The over-sampling scheme also begins with sampling the process data at a high frequency. The difference is that, then they identify a high frequency model from the high frequency sampled data without filtering and down-sampling the data; and they get the desired model by converting the high frequency model to the original low frequency. Wang et al. (2004b) gives solutions to the identification of multirate sampled-data system where the input and output are sampled at different sampling rates, this can be viewed as extension of the over-sampling problem. We are curious about the over-sampling scheme, can it achieve anti-aliasing by also using high-frequency sampling, or even do better than the anti-aliasing filtering procedure? And also, we would like to raise a fundamental question, can higher sampling frequency help increase model quality?

In Sun and co-workers' work (Sun and Sano, 2009; Sun and Zhu, 2012; Wang et al., 2004a), the over-sampling scheme is used in closed-loop system identification with no external excitation applied. They have proved, in different ways, that the

☆ This work is supported by the National Science Foundation of China (No. 61673343).

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http://dx.doi.org/10.1016/j.compchemeng.2017.07.010

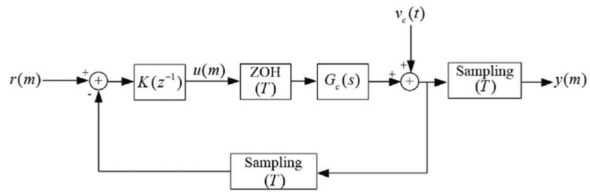


**Fig. 1.** A typical closed-loop system.

over-sampling scheme can achieve informative closed-loop identification tests even when there is no external excitation. However, they have not given the exact conditions on which the over-sampling scheme can work, while Wang et al. (2004a) points out that the performance of the over-sampling scheme is sensitive to many factors and sometimes the estimated model is of poor quality. Actually, external test signals are permitted in industrial applications, although they should be kept small in amplitudes. Therefore, in order to achieve high model quality and low disturbing test, we propose to use the over-sampling scheme with external excitation. We will further analyze the over-sampling scheme by using the asymptotic theory (Ljung, 1985) and propose an identification method based on the result. It will be surprisingly found that, the proposed method is a way that can both achieve anti-aliasing and make use of the high frequency noise as a part of excitation. We will also give the exact condition on which the good performance can be expected. And we point out it is also useful to apply the over-sampling scheme in open-loop identification.

Moreover, we would like to mention that the over-sampling operation is practically costless. In industrial MPC control systems, the controller sampling frequency is much lower than the sampling frequency of the lower level DCS (distributed control system) or PLC (programmable logic controller) systems. For example, in the refining/petrochemical industry, the sampling period of DCS systems is 1 s or smaller, while the sampling period of a typical MPC control system is 60 s. This means that in MPC identification, the sampling frequency can be made 60 times as high if necessary. The situations in other industries are similar. The redundancy of control systems sampling power is the foundation of the over-sampling scheme, but to the best of our knowledge, it has rarely been explored in the identification community.

The outline is as follows. Section 2 gives the problem statement. The anti-aliasing filtering procedure is described and analyzed in Section 3. In Section 4, the over-sampling scheme is introduced and further analyzed in frequency domain. The identification method based on the over-sampling scheme is proposed in Section 5. In Section 6, we compare the proposed method with the conventional identification. Some numerical examples are given in Section 7 to illustrate the results. Section 8 contains the conclusion.

## 2. Problem statement

In this work, we consider the identification of a closed-loop system depicted in Fig. 1. This system contains a linear continuous-time plant $G_c(s)$ and a linear digital controller $K(z^{-1})$. The control period is $T$, so is the sampling period in the feedback loop. The discrete-time control input $\{u(m)\}_{m=0,1,2,...}$ is zero-order-held with holding time $T$ before applying to the continuous-time plant. $\{r(m)\}_{m=0,1,2,...}$ is the reference signal with sample time $T$. $\{v_c(t)\}_{t\geq 0}$ is the continuous-time output noise.

The identification problem is to estimate the discretized model of $G_c(s)$ with respect to the sampling time $T$, denoted as $G_T(z^{-1})$. This model will be used in the following model based controller design. Although the problem statement is given with a closed-loop system, open-loop identification is also covered in our framework.

The model structure we consider is

$$G_T(z^{-1}) = \frac{B_T(z^{-1})}{A_T(z^{-1})} = \frac{b_{\tau_b} z^{-\tau_b} + \cdots + b_{n_b} z^{-n_b}}{1 + a_1 z^{-1} + \cdots + a_{n_a} z^{-n_a}} \tag{1}$$

where $n_a$ and $n_b$ are the orders of $A_T(z^{-1})$ and $B_T(z^{-1})$, $\tau_b$ is the delay.

The main assumptions through this paper are listed here.

**Assumption 1** *(True process).* The true model is contained in the model set defined by (1). The order of the true model and its delay are known as a priori. The delay $\tau_b \geq 1$.

**Assumption 2** *(Noise).* The output noise $v_c(t)$ is a stationary stochastic signal and the sampled signals from it can be described by stable and inversely stable discrete-time transfer functions.

**Assumption 3** *(Input).* In closed-loop identification, the reference signal is independent of the output noise; in open-loop case, the input signal is independent of the output noise.

## 3. Anti-aliasing filtering

### 3.1. General

To do identification in the closed-loop system in Fig. 1, first an identification experiment is conducted: the external excitation is applied (generally on the reference signal $r(m)$); the input and output are observed for the model estimation. Conventionally, the input and output are sampled at sampling period $T$ because we need to estimate the model $G_T(z^{-1})$. In Fig. 1, the control input $u(m)$ can be used as the $T$-sampled input. The $T$-sampled output is denoted as $\{y(m)\}_{m=0,1,2,...}$. $u(m)$ and $y(m)$ are then used to estimate the model $G_T(z^{-1})$:

$$y(m) = G_T(z^{-1})u(m) + v(m), \tag{2}$$

where $v(m)$ is the $T$-sampled signal from the continuous-time output noise $v_c(t)$.

However, there is a problem when sampling the data in this way: when the output noise $v_c(t)$ contains high frequency term outside the frequency band $[-\pi/T, \pi/T]$, the $T$-sampled output will contain aliasing from the high frequency noise, following from the Nyquist-Shannon sampling theorem. The noise aliasing will increase the model error in identification.

The anti-aliasing filtering procedure is aimed to avoid this kind of high frequency noise aliasing. When the following assumption holds,

**Assumption 4.** The spectrum of the output noise $v_c(t)$ is band limited and there exists a positive integer $p_0$ that the output noise $v_c(t)$ contains no frequency term outside the frequency band $[-p_0\pi/T, p_0\pi/T]$.

The procedure goes in three steps:

- Sample the input and output at a short enough sampling period $\Delta = T/p$, i.e. $p$ is a positive integer and $p > p_0$. Denote the $\Delta$-sampled input and output by $\{u_\Delta(k)\}_{k=0,1,2,...}$ and $\{y_\Delta(k)\}_{k=0,1,2,...}$, then $y_\Delta(k)$ will not have high frequency noise aliasing according to Assumption 4.
- Filter the $\Delta$-sampled data using a low-pass filter $F(z^{-1})$, of which the pass band is $[-\pi/p, \pi/p]$. Denote the filtered data by $\{u^f_\Delta(k)\}_{k=0,1,2,...}$ and $\{y^f_\Delta(k)\}_{k=0,1,2,...}$:

$$u^f_\Delta(k) = F(z^{-1})u_\Delta(k), \quad y^f_\Delta(k) = F(z^{-1})y_\Delta(k); \tag{3}$$

- Down-sample the filtered $\Delta$-sampled data at interval $p$. Since $T = p\Delta$, the sample time of the data generated in this step is $T$. Denote these data by $\{u^f(m)\}_{m=0,1,2,...}$ and $\{y^f(m)\}_{m=0,1,2,...}$:

$$u^f(m) = u^f_\Delta(pm), \quad y^f(m) = y^f_\Delta(pm). \tag{4}$$

Then $u^f(m)$ and $y^f(m)$ are used in the model estimation instead of $u(m)$ and $y(m)$. Denote the estimated model by $G^f_T(z^{-1})$:

$$y^f(m) = G^f_T(z^{-1})u^f(m) + v^f(m), \tag{5}$$

where $v^f(m)$ is the down-sampled filtered output noise. Now we question that if $G^f_T(z^{-1})$ is a consistent estimate of $G_T(z^{-1})$, the answer is no, and the analysis will be given in the following two subsections.

### 3.2. Time-domain analysis using lifting technique

First we use lifting technique to analysis the anti-aliasing filtering in time-domain. Lifting technique is an operation to convert a 1-dimensional signal into a multi-dimensional signal (see Chen and Francis, 2012). For example, lifting the $\Delta$-sampled input $u_\Delta(k)$ in rate $p$, you get a $p$-dimensional signal

$$\underline{u_\Delta} := \left\{ \begin{bmatrix} u_\Delta(0) \\ u_\Delta(1) \\ \vdots \\ u_\Delta(p-1) \end{bmatrix}, \begin{bmatrix} u_\Delta(p) \\ u_\Delta(p+1) \\ \vdots \\ u_\Delta(2p-1) \end{bmatrix}, \cdots \right\}, \tag{6}$$

where the underline denotes that the signal is lifted. We lift all the signals with sample time $\Delta$ in the same way, and denote the lifted $\Delta$-sampled signals by $\underline{u_\Delta}$ and $\underline{y_\Delta}$, and the lifted filtered $\Delta$-sampled signals by $\underline{u^f_\Delta}$ and $\underline{y^f_\Delta}$.

Now we start from the transfer function between the $\Delta$-sampled input and output, which is a discrete-time model with respect to sampling period $\Delta$ and denoted by $G_\Delta(z^{-1})$:

$$y_\Delta(k) = G_\Delta(z^{-1})u_\Delta(k) + v_\Delta(k), \tag{7}$$

where $v_\Delta(k)$ is the $\Delta$-sampled output noise. After doing the lifting operation, we also have a transfer function between the lifted signals $\underline{u_\Delta}$ and $\underline{y_\Delta}$, denoted as $\underline{G_\Delta}(z^{-1})$:

$$\underline{y_\Delta} = \underline{G_\Delta}(z^{-1})\underline{u_\Delta} + \underline{v_\Delta}, \tag{8}$$

where $\underline{v_\Delta}$ is the lifted $\Delta$-sampled output noise. Observing the structure in (6), we find that each lifted signal can be taken as a vector containing $p$ signals, so $\underline{G_\Delta}(z^{-1})$ can be taken as a $p$-input $p$-output transfer function. Denote the state-space realization of $G_\Delta(z^{-1})$ by

$$G_\Delta(z^{-1}) = \left[\begin{array}{c|c} A_\Delta & B_\Delta \\ \hline C_\Delta & D_\Delta \end{array}\right], \tag{9}$$

then according to Theorem 8.2.1 in Chen and Francis (2012), the state-space realization of $\underline{G_\Delta}(z^{-1})$ can be represented by

$$\underline{G_\Delta}(z^{-1}) = \left[\begin{array}{c|cccc} A^p & A^{p-1}B & A^{p-2}B & \cdots & B \\ \hline C & D & 0 & \cdots & 0 \\ CA & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ CA^{p-1} & CA^{p-2}B & CA^{p-3}B & \cdots & D \end{array}\right]. \tag{10}$$

Here and in the sequel, the subscript $\Delta$ is left out in $A_\Delta, B_\Delta, C_\Delta$ and $D_\Delta$ to reduce the notational burden without risk of confusion.

Since we have $\{y(m) = y_\Delta(pm)\}_{m=0,1,2,\ldots}$, $y(m)$ is thus the first output of the $p$-input $p$-output transfer function $\underline{G_\Delta}(z^{-1})$. So following from (10), there is

$$y(m) = \sum_{j=0}^{p-1} g_j(z^{-1})u_\Delta(mp+j) + v(m) \tag{11}$$

where

$$g_0(z^{-1}) = \left[\begin{array}{c|c} A^p & A^{p-1}B \\ \hline C & D \end{array}\right], \quad g_{j\neq 0}(z^{-1}) = \left[\begin{array}{c|c} A^p & A^{p-1-j}B \\ \hline C & 0 \end{array}\right]. \tag{12}$$

Moreover, due to the zero-order holding, we have

$$u_\Delta(mp+j) = u(m), j = 0, 1, \cdots, p-1, \tag{13}$$

so (11) can also be written as

$$y(m) = \sum_{j=0}^{p-1} g_j(z^{-1})u(m) + v(m). \tag{14}$$

Comparing (14) with (2), we get to know the relation between $G_T(z^{-1})$ and $G_\Delta(z^{-1})$ in time-domain, that is

$$G_T(z^{-1}) = \sum_{j=0}^{p-1} g_j(z^{-1}). \tag{15}$$

On the other hand, after filtering the $\Delta$-sampled data, we still have $G_\Delta(z^{-1})$ as the transfer function between the filtered data, and $\underline{G_\Delta}(z^{-1})$ as the transfer function between the lifted filtered data:

$$\begin{aligned} y^f_\Delta(k) &= G_\Delta(z^{-1})u^f_\Delta(k) + v^f_\Delta(k), \\ \underline{y^f_\Delta} &= \underline{G_\Delta}(z^{-1})\underline{u^f_\Delta} + \underline{v^f_\Delta}, \end{aligned} \tag{16}$$

where $v^f_\Delta(k)$ is the filtered $\Delta$-sampled output noise, $\underline{v^f_\Delta}$ is its lifted version.

In the same methodology as discussed with (11), we have

$$y^f(m) = \sum_{j=0}^{p-1} g_j(z^{-1})u^f_\Delta(mp+j) + v^f(m) \tag{17}$$

where $g_j(z^{-1})$ is (12).

However, after doing the filtering, the zero-order holding no longer exists, here we only have $u^f_\Delta(mp) = u^f(m)$. But the relation between $\{u^f_\Delta(mp+j)\}_{j=1,\ldots,p-1}$, and $u^f(m)$ can be obtained through the filtering operation (3). In the same manner as (8), we have that the $p$-input $p$-output transfer function between the lifted signal $\underline{u}^f_\Delta$ and $\underline{u}_\Delta$ is

$$\underline{u}^f_\Delta = \underline{F}(z^{-1})\underline{u}_\Delta. \tag{18}$$

And if we denote the state-space matrix of the filter $F(z^{-1})$ as $A_f$, $B_f$, $C_f$ and $D_f$, then similarly as (10) we have

$$\underline{F}(z^{-1}) = \left[\begin{array}{c|cccc} A_f^p & A_f^{p-1}B_f & A_f^{p-2}B_f & \cdots & B_f \\ \hline C_f & D_f & 0 & \cdots & 0 \\ C_fA_f & C_fB_f & D_f & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C_fA_f^{p-1} & C_fA_f^{p-2}B_f & C_fA_f^{p-3}B_f & \cdots & D_f \end{array}\right]. \tag{19}$$

We know that $u^f_\Delta(mp+j)$ is the $(j+1)$th output of $\underline{F}(z^{-1})$, by taking into account the zero-order holding (13), we have

$$u^f_\Delta(mp+j) = H_j(z^{-1})u(m) \tag{20}$$

where

$$H_0(z^{-1}) = \left[\begin{array}{c|c} A_f^p & A_f^{p-1}B_f + A_f^{p-2}B_f + \cdots + B_f \\ \hline C_f & D_f \end{array}\right]$$
$$H_{j\neq 0}(z^{-1}) = \left[\begin{array}{c|c} A_f^p & A_f^{p-1}B_f + A_f^{p-2}B_f + \cdots + B_f \\ \hline C_fA_f^j & C_fA_f^{j-1}B_f + \cdots + D_f \end{array}\right]. \tag{21}$$

So for $j = 1, \ldots, p-1$, there is

$$u^f_\Delta(mp+j) = \frac{H_j(z^{-1})}{H_0(z^{-1})}u^f_\Delta(mp) = \frac{H_j(z^{-1})}{H_0(z^{-1})}u^f(m), \tag{22}$$

plugging (22) into (17), then we have the relation between $G^f_T(z^{-1})$ and $G_\Delta(z^{-1})$:

$$G^f_T(z^{-1}) = \sum_{j=0}^{p-1}\frac{H_j(z^{-1})}{H_0(z^{-1})}g_j(z^{-1}) \tag{23}$$

Comparing (23) with (15), we then observe that $G^f_T(z^{-1})$ is not equal to $G_T(z^{-1})$ in general, and due to the filtering, $G^f_T(z^{-1})$ has higher order than $G_T(z^{-1})$.

### 3.3. Frequency-domain analysis

In this subsection, we will further analyze the anti-aliasing filtering in frequency domain. Let the continuous-time system $G_c(s)$ be a causal stable linear time-invariant system (this is implied in Assumption 1) with frequency function $G_c(i\omega)$, then first we want to relate the frequency responses of the corresponding sampled system for two different sampling periods $T$ and $\Delta$.

Let $u_c^\Delta(t)$ be a continuous-time pulse signal with magnitude 1, and width $\Delta$, starting at time $t=0$. Denote the output of $G_c(s)$ with this input by $\bar{g}_c^\Delta(t)$, and the Fourier transform of this output by $\bar{G}_c^\Delta(i\omega)$. Sampling $\bar{g}_c^\Delta(t)$ with sampling period $\Delta$, we can get the frequency response of the sampled system $G_\Delta(z^{-1})$, that is

$$G_\Delta(e^{i\omega}) = \frac{1}{\Delta}\sum_{k=-\infty}^{\infty}\bar{G}_c^\Delta\left(i\frac{\omega-2\pi k}{\Delta}\right). \tag{24}$$

Now consider $u_c^T(t)$ as a continuous-time pulse signal with magnitude 1, and width $T$, starting at time $t=0$. Then $u_c^T(t)$ can be constructed from $u_c^\Delta(t)$ as

$$u_c^T(t) = \sum_{l=0}^{p-1}u_c^\Delta(t-\Delta l). \tag{25}$$

Then, due to the linearity and time-invariance of $G_c(s)$, the output of $G_c(s)$ with this input, denoted by $\bar{g}_c^T(t)$, has the Fourier transform

$$\bar{G}_c^T(i\omega) = \sum_{l=0}^{p-1}e^{-i\omega\Delta l}\bar{G}_c^\Delta(i\omega). \tag{26}$$

Sampling $\bar{g}_c^T(t)$ with the sampling period $T$, we get the frequency response of the sampled system $G_T(z^{-1})$

$$\begin{aligned} G_T(e^{i\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\bar{G}_c^T\left(i\frac{\omega-2\pi k}{T}\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{l=0}^{p-1}e^{-i\Delta l\frac{\omega-2\pi k}{T}}\bar{G}_c^\Delta\left(i\frac{\omega-2\pi k}{T}\right) = \{k=pm+r\} \\ &= \frac{1}{T}\sum_{m=-\infty}^{\infty}\sum_{r=0}^{p-1}\sum_{l=0}^{p-1}e^{-i\Delta l\frac{\omega-2\pi(pm+r)}{T}}\bar{G}_c^\Delta\left(i\frac{\omega-2\pi(pm+r)}{T}\right) \\ &= \frac{1}{T}\sum_{r=0}^{p-1}\sum_{l=0}^{p-1}e^{-il\frac{\omega-2\pi r}{p}}\sum_{m=-\infty}^{\infty}\bar{G}_c^\Delta\left(i\frac{\omega-2\pi r}{p\Delta}-\frac{2\pi m}{\Delta}\right) \\ &= \frac{1}{p}\sum_{r=0}^{p-1}\sum_{l=0}^{p-1}e^{-il\frac{\omega-2\pi r}{p}}G_\Delta\left(e^{i\frac{\omega-2\pi r}{p}}\right) \\ &= \frac{1}{p}\sum_{r=0}^{p-1}G_\Delta\left(e^{i\frac{\omega-2\pi r}{p}}\right)f_p\left(\frac{\omega-2\pi r}{p}\right), \end{aligned} \tag{27}$$

the last equality is with

$$f_p(\omega) := \sum_{l=0}^{p-1}e^{-il\omega}. \tag{28}$$

So (27) is the relation between $G_\Delta(e^{i\omega})$ and $G_T(e^{i\omega})$.

We then consider the system after the anti-aliasing filtering is applied. Denote the frequency response of the filter $F(z^{-1})$ by $F(e^{i\omega})$, and the corresponding Discrete Fourier transform of the filtered $\Delta$-sampled data in (3) by $U_\Delta^F(e^{i\omega})$ and $Y_\Delta^F(e^{i\omega})$. This filtered data is then down-sampled to sampling period $T$ in (4). We denote the corresponding Discrete Fourier transforms of the down-sampled signals by $U_T^F(e^{i\omega})$ and $Y_T^F(e^{i\omega})$. Then we have

$$\begin{aligned} U_T^F(e^{i\omega}) &= \frac{1}{p}\sum_{l=0}^{p-1} U_\Delta^F\left(e^{i\frac{\omega-2\pi l}{p}}\right) \\ &= \frac{1}{p}\sum_{l=0}^{p-1} F\left(e^{i\frac{\omega-2\pi l}{p}}\right) U_\Delta\left(e^{i\frac{\omega-2\pi l}{p}}\right) \\ &= \frac{1}{p}\sum_{l=0}^{p-1} F\left(e^{i\frac{\omega-2\pi l}{p}}\right) f_p\left(\frac{\omega-2\pi l}{p}\right) U_T\left(e^{i\frac{\omega-2\pi l}{p}p}\right) \\ &= \left(\frac{1}{p}\sum_{l=0}^{p-1} F\left(e^{i\frac{\omega-2\pi l}{p}}\right) f_p\left(\frac{\omega-2\pi l}{p}\right)\right) U_T(e^{i\omega}). \end{aligned} \tag{29}$$

Here the first equality follows from the standard result about down-sampling a discrete-time signal

$$\begin{aligned} U_T^F(e^{i\omega}) &= \sum_{m=-\infty}^{\infty} u^f(m)e^{-i\omega m} \\ &= \sum_{m=-\infty}^{\infty} u_\Delta^f(mp)e^{-i\omega m} = \left\{\frac{1}{p}\sum_{l=0}^{p-1} e^{i\frac{2\pi}{p}kl} = \sum_{m=-\infty}^{\infty}\delta(k-mp)\right\} \\ &= \sum_{k=-\infty}^{\infty} u_\Delta^f(k)\left(\frac{1}{p}\sum_{l=0}^{p-1} e^{i\frac{2\pi}{p}kl}\right)e^{-i\omega\frac{k}{p}} \\ &= \frac{1}{p}\sum_{l=0}^{p-1}\sum_{k=-\infty}^{\infty} u_\Delta^f(k)e^{i\frac{2\pi}{p}kl}e^{-i\omega\frac{k}{p}} \\ &= \frac{1}{p}\sum_{l=0}^{p-1} U_\Delta^F\left(e^{i\frac{\omega-2\pi l}{p}}\right), \end{aligned} \tag{30}$$

where $\delta(k)$ is the Dirac function. The third equality in (29) follows from the zero-order holding of $u_\Delta(k)$

$$\begin{aligned} U_\Delta(e^{i\omega}) &= \sum_{k=-\infty}^{\infty} u_\Delta(k)e^{-i\omega k} = \{k = mp + l\} \\ &= \sum_{m=-\infty}^{\infty}\sum_{l=0}^{p-1} u_\Delta(pm+l)e^{-i\omega(pm+l)} \\ &= \sum_{l=0}^{p-1} e^{-i\omega l}\sum_{m=-\infty}^{\infty} u(m)e^{-i\omega pm} \\ &= f_p(\omega)U_T(e^{i\omega p}) \end{aligned} \tag{31}$$

Similarly we have

$$\begin{aligned} Y_T^F(e^{i\omega}) &= \frac{1}{p}\sum_{l=0}^{p-1} Y_\Delta^F\left(e^{i\frac{\omega-2\pi l}{p}}\right) \\ &= \frac{1}{p}\sum_{l=0}^{p-1} F\left(e^{i\frac{\omega-2\pi l}{p}}\right) Y_\Delta\left(e^{i\frac{\omega-2\pi l}{p}}\right) \\ &= \frac{1}{p}\sum_{l=0}^{p-1} F\left(e^{i\frac{\omega-2\pi l}{p}}\right) G_\Delta\left(e^{i\frac{\omega-2\pi l}{p}}\right) U_\Delta\left(e^{i\frac{\omega-2\pi l}{p}}\right) \\ &= \left(\frac{1}{p}\sum_{l=0}^{p-1} F\left(e^{i\frac{\omega-2\pi l}{p}}\right) f_p\left(\frac{\omega-2\pi l}{p}\right) G_\Delta\left(e^{i\frac{\omega-2\pi l}{p}}\right)\right) U_T(e^{i\omega}) \end{aligned} \tag{32}$$

Then we can get the frequency response of the transfer function $G_T^f(z^{-1})$

$$G_T^F(e^{i\omega}) := \frac{Y_T^F(e^{i\omega})}{U_T^F(e^{i\omega})} = \frac{\sum_{l=0}^{p-1} F\left(e^{i\frac{\omega-2\pi l}{p}}\right) f_p\left(\frac{\omega-2\pi l}{p}\right) G_\Delta\left(e^{i\frac{\omega-2\pi l}{p}}\right)}{\sum_{l=0}^{p-1} F\left(e^{i\frac{\omega-2\pi l}{p}}\right) f_p\left(\frac{\omega-2\pi l}{p}\right)} \tag{33}$$

This relationship is in general not equal to $G_T(e^{i\omega})$ (27) due to the presence of $F(e^{i\omega})$. We then consider the case when $F(z^{-1})$ is an ideal low-pass filter, i.e. $F(e^{i\omega})$ is zero for $\omega \in (\frac{\pi}{p}, \quad \pi]$. Then (33) reduces to

$$G_T^F(e^{i\omega}) = G_\Delta(e^{i\frac{\omega}{p}}), \text{ in } \quad \omega \in \left[-\frac{\pi}{p}, \frac{\pi}{p}\right]. \tag{34}$$

Still it is not consistent with $G_T(e^{i\omega})$, but it is approximate to $G_T(e^{i\omega})$ (27) when $G_\Delta(e^{i\omega})$ is insignificant outside the frequency band $[-\pi/p, \pi/p]$, i.e. $G_c(i\omega)$ is insignificant outside $[-\pi/T, \pi/T]$.

Above all, we can say that, the anti-aliasing filtering procedure will increase the order of the underlying system (from time-domain analysis) and does not give a consistent estimate of $G_T(e^{i\omega})$ (from both time-domain and frequency domain analysis). When an ideal low-pass filter is used, the procedure gives an approximate estimate on condition that the continuous-time system is insignificant outside the frequency band $[-\pi/T, \pi/T]$.

## 4. Over-sampling scheme

From the analysis in Section 3, we know that the anti-aliasing filtering procedure is not an optimal way to solve the anti-aliasing problem. In this section, we propose the use of the over-sampling scheme (Sun and Sano, 2009), and we will explain in the next section why the over-sampling scheme can solve the anti-aliasing problem and gain extra accuracy.

Same as the anti-aliasing filtering procedure, the over-sampling scheme also starts with sampling the process data using a short enough sampling period $\Delta = T/p$, this is why it is called "over-sampling". The difference is that, after the first step, instead of filtering the $\Delta$-sampled data and down-sampling it, we directly estimate the discrete-time $\Delta$-model $G_\Delta(z^{-1})$ in (7) from $u_\Delta(k)$ and $y_\Delta(k)$. Then we use model conversion to get the $T$-model $G_T(z^{-1})$. The model conversion can be conducted using (15) with (12), or following the conversion in Sun et al. (1997).

### 4.1. Frequency domain properties

In this subsection, we analysis the properties of the over-sampling scheme in frequency domain. We apply the asymptotic theory (Ljung, 1985), and derive an explicit asymptotic variance expression for the $\Delta$-model estimated in the over-sampling scheme, this variance expression is asymptotic both in the data length and in the model order.

We first consider the frequency domain properties of the input and output. Due to the zero-order holding, the over-sampled input and output, i.e. $u_\Delta(k)$ and $y_\Delta(k)$, are cyclo-stationary signals, for which the cyclo-stationary correlation function is defined as

$$C_{x_1,x_2}(\alpha_l, \tau) = \frac{1}{p}\sum_{k=mp}^{mp+p-1} e^{i\alpha_l k}E\{x_1(k+\tau)x_2(k)\} \tag{35}$$

where either $x_1(k)$ or $x_2(k)$ is cyclo-stationary signal, $m$ is an arbitrary integer, $\alpha_l = 2\pi l/p$, $l = 0, 1, \ldots, p-1$. The cyclo-stationary spectrum is further defined as

$$S_{x_1,x_2}(\alpha_l,\omega)=\sum_{\tau=-\infty}^{\infty}C_{x_1,x_2}(\alpha_l,\tau)e^{-i\omega\tau}. \tag{36}$$

See Sun and Sano (2009) for more details about this part.

Then there is the following lemma about the cyclo-stationary spectrum of $u_\Delta(k)$.

**Lemma 1.** *The relation between the cyclo-stationary spectrum of $u_\Delta(k)$ and the spectrum of $u(m)$ can be represented by*

$$S_{u_\Delta,u_\Delta}(\alpha_l,\omega)=\frac{1}{p}\sum_{j=0}^{p-1}e^{-ij\omega}\sum_{j=0}^{p-1}e^{ij(\omega-\alpha_l)}\Phi_u(p\omega), \tag{37}$$

*where $\Phi_u(\omega)$ is the spectrum of $u(m)$.*

$$v_\Delta(k)=H_\Delta(z^{-1})e_\Delta(k), \tag{38}$$

*where $e_\Delta(k)$ is a Gaussian white noise with zero mean and variance $\lambda_\Delta$. Then there is*

$$S_{u_\Delta,e_\Delta}(\alpha_l,\omega)=\frac{\lambda_\Delta}{p}\sum_{j=0}^{p-1}e^{-ij\omega}\frac{-K(e^{ip\omega})H_\Delta(e^{-i(\omega-\alpha_l)})}{1+K(e^{ip\omega})G_T(e^{ip\omega})}, \tag{39}$$

*where $K(e^{i\omega})$ and $H_\Delta(e^{i\omega})$ are the frequency responses of $K(z^{-1})$ and $H_\Delta(z^{-1})$.*

**Proof.** See Appendix A. □

Before giving the asymptotic variance expression, we first define the following function

$$\Lambda(\omega,p):=\frac{1}{p}\sum_{j=0}^{p-1}e^{-ij\omega}\sum_{j=0}^{p-1}e^{ij\omega}. \tag{40}$$

Then we have the following theorem.

**Theorem 1.** *Let the Assumptions 1–3 be in force, consider the Δ-model estimated with order n in the over-sampling scheme. When the data length N and the model order n both turn to infinity, the estimated model $\hat{G}_\Delta(z^{-1},n)$ is asymptotically consistent and follows a Gaussian distribution. In open-loop identification, the asymptotic variance (asymptotic both in the data length N and the model order n) of $\hat{G}_\Delta(z^{-1},n)$ in frequency domain is*

$$\text{AsVar}\quad\hat{G}_\Delta(e^{i\omega},n)\sim\frac{n}{N}\frac{\Phi_{v_\Delta}(\omega)}{\Lambda(\omega,p)\Phi_u(p\omega)}, \tag{41}$$

*where $\Phi_{v_\Delta}(\omega)$ is the spectrum of $v_\Delta(k)$. In closed-loop identification, the asymptotic variance is*

$$\text{AsVar}\quad\hat{G}_\Delta(e^{i\omega},n)\sim\frac{n}{N}\frac{\Phi_{v_\Delta}(\omega)}{\Lambda(\omega,p)(S_r(\omega)+S_e(\omega))}, \tag{42}$$

*where*

$$S_r(\omega)=\left|\frac{K(e^{ip\omega})}{1+K(e^{ip\omega})G_T(e^{ip\omega})}\right|^2\Phi_r(p\omega)$$

$$S_e(\omega)=\left|\frac{K(e^{ip\omega})}{1+K(e^{ip\omega})G_T(e^{ip\omega})}\right|^2\left(\Phi_v(p\omega)-\frac{1}{p}\Phi_{v_\Delta}(\omega)\right),$$

*and $\Phi_r(\omega)$ is the spectrum of $r(m)$, $\Phi_v(\omega)$ is the spectrum of $v(m)$.*

**Proof.** The asymptotic result for the over-sampling based identification has been derived in Fang and Zhu (2017) under the assumptions, by applying the asymptotic theory in Ljung (1985). The theorem here focuses on the explicit asymptotic variance expression. The proof follows from Theorem 1 in Fang and Zhu (2017), according to which the asymptotic variance of $\hat{G}_\Delta(e^{i\omega},n)$ is

$$\text{AsVar}\quad\hat{G}_\Delta(e^{i\omega},n)\sim\frac{n}{N}\frac{\lambda_\Delta\Phi_{v_\Delta}(\omega)}{\lambda_\Delta S_{u_\Delta,u_\Delta}(0,\omega)-S_{u_\Delta,e_\Delta}(0,\omega)S_{e_\Delta,u_\Delta}(0,\omega)}. \tag{43}$$

Then (41) and (42) come directly by applying Lemma 1 and (40). □

Now we talk more about the function $\Lambda(\omega,p)$ that appears in (41) and (42) ($\Lambda(\omega,p)$ when $p=4$ is plotted in Fig. 2 as an example). We observe that $\Lambda(0,p)=p$ and that it can be thought of as a low-pass function with pass band $[-\pi/p,\pi/p]$. $\Lambda(\omega,p)$ appears in the numerator of the asymptotic variance expression, this implies that, when we have the estimate $\hat{G}_\Delta(e^{i\omega},n)$, its information contained in the lower frequency band $[-\pi/p,\pi/p]$ is more reliable, i.e. with smaller variance, than that in the higher frequency band where the excitation is “filtered out” by $\Lambda(\omega,p)$. This is what we need to address in the identification method.

**Remark 1.** In the next section, we will take the approximation $\Lambda(\omega,p)\approx p$ in $\omega\in[-\pi/p,\pi/p]$, this is reasonable because the asymptotic variance of $\hat{G}_\Delta(e^{i\omega},n)$ is generally low-pass, following from the fact that the model itself is generally low-pass in practice. Then it makes sense that we take the value $\Lambda(0,p)$ to count for its influence in the model variance.

## 5. Identification method

In this section, we propose an identification method based on the above described over-sampling scheme. We find that, the most natural way to use the asymptotic variance expression derived in Section 4 and deal with the above mentioned low-pass property of $\Lambda(\omega,p)$ is to modify the so-called asymptotic method (ASYM) (Zhu, 1998), as this method is based on the asymptotic theory and use the asymptotic variance expression inside.

The new identification method is referred to as ASYM$_\text{p}$, it goes in three steps as follows and we will also explain how we modify ASYM.

- Estimate a high-order ARX model $\hat{G}^n_\Delta(z^{-1},n)$ using $u_\Delta(k)$ and $y_\Delta(k)$, that is

$$\begin{aligned}&\hat{A}_\Delta(z^{-1},n)y_\Delta(k)=\hat{B}_\Delta(z^{-1},n)u_\Delta(k)+\hat{e}_\Delta(k)\\&\hat{G}_\Delta(z^{-1},n)=\frac{\hat{B}(z^{-1},n)}{\hat{A}(z^{-1},n)};\end{aligned} \tag{44}$$

- Based on Theorem 1, the asymptotic variance of $\hat{G}_\Delta(z^{-1},n)$ in frequency domain is given in (41) and (42). Treat $\hat{G}_\Delta(e^{i\omega},n)$ as the observation of the plant (the frequency domain data), then one can apply the maximum likelihood method to estimate the reduced model, i.e. the model with true order. It can be shown that the negative log-likelihood function for the reduced model is given by

$$V=\int_{-\frac{\pi}{p}}^{\frac{\pi}{p}}|\hat{G}_\Delta(e^{i\omega},n)-\hat{G}_\Delta(e^{i\omega})|^2(AsVar\quad\hat{G}_\Delta(e^{i\omega},n))^{-1}d\omega, \tag{45}$$

where $\hat{G}_\Delta(e^{i\omega})$ is the reduced model and is estimated by minimizing (45) for the true model order. Note that the integration interval we use here is $[-\frac{\pi}{p},\frac{\pi}{p}]$, while the integration interval in the normal ASYM is $[-\pi,\pi]$. In this way, we only take into account the information contained in $\hat{G}_\Delta(e^{i\omega},n)$ in the pass band of $\Lambda(\omega,p)$.

- Convert the Δ-model $\hat{G}_\Delta(z^{-1})$ to the $T$-model $\hat{G}_T(z^{-1})$.

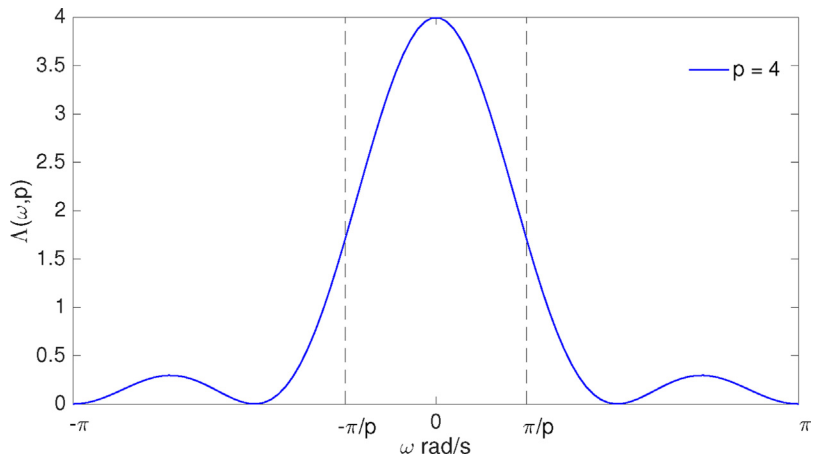


**Fig. 2.** The function $\Lambda(\omega, p)$ when $p = 4$.

**Remark 2.** Although the frequency domain properties of the over-sampling scheme is analyzed for models with infinitely high order, we conjecture that the result also holds for finite order models. In $\mathrm{ASYM_p}$, by using the integrate interval $\left[-\frac{\pi}{p}, \frac{\pi}{p}\right]$, we are eventually performing low-pass filtering in frequency domain to only focus on the information of $\hat{G}_\Delta(e^{i\omega}, n)$ in lower frequency band. One can also do the filtering in time-domain, i.e. use a low-pass filter $F(z^{-1})$ of which the pass band is $\left[-\frac{\pi}{p}, \frac{\pi}{p}\right]$ to filter $u_\Delta(k)$ and $y_\Delta(k)$, just as the second step of the anti-aliasing filtering procedure. Then one can estimate $\hat{G}_\Delta(z^{-1})$ from the filtered data using e.g. the prediction error method, then do model conversion. We observed in our simulations that this procedure always has the similar efficiency as $\mathrm{ASYM_p}$.

## 6. Comparison with conventional identification

Then we compare the proposed method with the conventional identification, where the process data is sampled with the sampling period $T$ and the model $G_T(z^{-1})$ is directly identified from the $T$-sampled data.

Here we consider the asymptotic variance of models with infinitely high order. In conventional identification, according to the asymptotic theory (Ljung, 1985), the asymptotic variance of the estimated model $\hat{G}_T(z^{-1}, n)$ in closed-loop identification is

$$\text{AsVar}\ \ \hat{G}_{con}(e^{i\omega}, n) \sim \frac{n}{N}\frac{\Phi_v(\omega)}{\left|\frac{K(e^{i\omega})}{1+K(e^{i\omega})G_T(e^{i\omega})}\right|^2 \Phi_r(\omega)}. \tag{46}$$

here we use the subscript *con* in short of "conventional".

Note that in the same identification experiment, when the process data is over-sampled at rate $p$, the data length becomes $pN$. Then according to Theorem 1, the asymptotic variance of $\hat{G}_\Delta(z^{-1}, n)$ estimated using the proposed method (in closed-loop identification) is

$$\text{AsVar}\ \ \hat{G}_\Delta(e^{i\omega}, n) \sim \frac{n}{pN}\frac{\Phi_{v_\Delta}(\omega)}{\Lambda(\omega, p)(S_r(\omega) + S_e(\omega))}. \tag{47}$$

However, the data length $pN$ does not imply a reduction in the variance of the final estimate. In Step 2 of $\mathrm{ASYM_p}$, only $1/p$ of the frequency domain information (with integrate interval $[-\pi/p, \pi/p]$) is taken into account when estimating the reduced model, this will make the variance of the reduced model multiplied by $p$, so overall the data length $pN$ does not effect the model variance when we consider the model with true order. Therefore, we can ignore the ratio $n/pN$ in (47) when compare it with the asymptotic variance in conventional identification.

Now we still cannot compare (47) and (46) directly, because they are for systems with different sample times. Recalling that we have the frequency domain relation (27) between the systems with sample time $T$ and $\Delta$ in Section 3.3. Observing (27), we find that when the sampling period $T$ is chosen small enough that the continuous-time process $G_c(s)$ contains little energy outside the frequency band $[-\pi/T, \pi/T]$, there is $G_T(e^{i\omega}) \approx G_\Delta(e^{i\omega/p})$. So the asymptotic variance of the model $\hat{G}_T(z^{-1})$ estimated with the proposed method (although the model $\hat{G}_T(z^{-1}, n)$ is not estimated in $\mathrm{ASYM_p}$, it still makes sense that we have an asymptotic variance for $\hat{G}_T(z^{-1})$ when the high order $n$ is used to estimate $\hat{G}_\Delta(z^{-1}, n)$, and the reduced model $\hat{G}_\Delta(z^{-1})$ is converted to $\hat{G}_T(z^{-1})$ with almost no conversion error) can be taken as

$$\text{AsVar}\ \ \hat{G}_{pro}(e^{i\omega}, n) \approx \text{AsVar}\ \ \hat{G}_\Delta(e^{i\frac{\omega}{p}}, n), \quad \omega \in [-\pi, \pi] \tag{48}$$

here the subscript *pro* is in short of "proposed".

It has been illustrated in Fang and Zhu (2017) that

$$\Phi_v(\omega) = \frac{1}{p}\sum_{k=0}^{p-1}\Phi_{v_\Delta}\left(\frac{\omega - 2\pi k}{p}\right), \tag{49}$$

with (49) and take $\Lambda(\omega/p, p) \approx p$, the asymptotic variance of $\hat{G}_T(z^{-1})$ estimated using the proposed method (in closed-loop identification) can be rewritten as

$$\text{AsVar}\ \ \hat{G}_{pro}(e^{i\omega}, n) \sim \frac{n}{pN}\frac{\Phi_v(\omega) - \frac{1}{p}\sum_{k=1}^{p-1}\Phi_{v_\Delta}\left(\frac{\omega-2\pi k}{p}\right)}{\left|\frac{K(e^{i\omega})}{1+K(e^{i\omega})G_T(e^{i\omega})}\right|^2\left(\Phi_r(\omega) + \frac{1}{p}\sum_{k=1}^{p-1}\Phi_{v_\Delta}\left(\frac{\omega-2\pi k}{p}\right)\right)}. \tag{50}$$

In this expression, the part $\frac{1}{p}\sum_{k=1}^{p-1}\Phi_{v_\Delta}\left(\frac{\omega-2\pi k}{p}\right)$ is the aliasing from the output noise beyond the bandwidth of the plant (the high frequency noise). Comparing (50) with (46), one can see two improvements using the proposed method: (a) the term $-\frac{1}{p}\sum_{k=1}^{p-1}\Phi_{v_\Delta}\left(\frac{\omega-2\pi k}{p}\right)$ in the numerator of (50) means that the high frequency noise aliasing is removed, which implies anti-aliasing; (b) the term $\frac{1}{p}\sum_{k=1}^{p-1}\Phi_{v_\Delta}\left(\frac{\omega-2\pi k}{p}\right)$ in the denominator of (50) acts as additional excitation.

In open-loop identification, similarly there is

$$\text{AsVar}\ \ \hat{G}_{con}(e^{i\omega}, n) \sim \frac{n}{N}\frac{\Phi_v(\omega)}{\Phi_u(\omega)}, \tag{51}$$

$$\text{AsVar}\ \ \hat{G}_{pro}(e^{i\omega}, n) \sim \frac{n}{pN}\frac{\Phi_v(\omega) - \frac{1}{p}\sum_{k=1}^{p-1}\Phi_{v_\Delta}\left(\frac{\omega-2\pi k}{p}\right)}{\Phi_u(\omega)}, \tag{52}$$

so the effect (a) also holds for open-loop tests.

Now we can answer the two questions raised in the abstract. Since effect (a) and effect (b) only exist with the high frequency output noise, so higher sampling frequency can help increase model

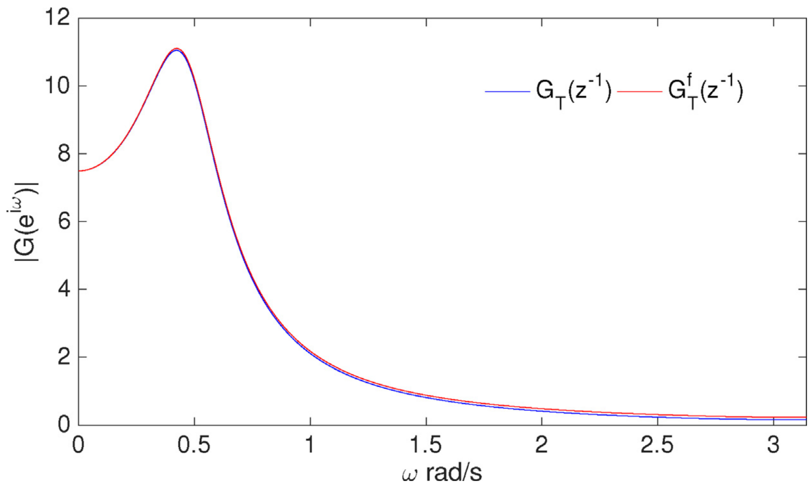


**Fig. 3.** The frequency responses of $G_T(z^{-1})$ and $G_T^f(z^{-1})$ (in magnitude).

accuracy only when the output noise contains energy beyond the bandwidth of the plant. And the proposed method can do better than the anti-aliasing filtering for it implies anti-aliasing in both open-loop and closed-loop tests while the anti-aliasing filtering procedure gives inconsistent estimate. Moreover, the proposed method can turn the high frequency noise into extra excitation in closed-loop tests.

## 7. Numerical simulations

### *7.1. Anti-aliasing filtering*

**Example 1.** Consider a closed-loop control system depicted in Fig. 1 with the continuous time process

$$G_c(s) = \frac{0.254s + 1.8198}{s^2 + 0.3567s + 0.2426}, \quad (53)$$

and the controller

$$K(z^{-1}) = \frac{1.5 - 0.7z^{-1}}{1 + 0.5z^{-1}}, \quad (54)$$

the control period is $T = 1$. Thus the true model is

$$G_T(z^{-1}) = \frac{z^{-1} + 0.5z^{-2}}{1 - 1.5z^{-1} + 0.7z^{-2}}. \quad (55)$$

In the simulation, no output noise is applied in order to test the consistency of the anti-aliasing filtering procedure. The reference signal $r(m)$ is a GBN (general binary noise) signal with average switching time 10. $p$ is chosen as $p = 2$ and the low-pass filter $F(z^{-1})$ is

$$F(z^{-1}) = \frac{0.3 + 0.6z^{-1} + 0.3z^{-2}}{1 + 0.19z^{-1} + 0.17z^{-1}}. \quad (56)$$

Then we can calculate the model estimated in the anti-aliasing filtering procedure by (23) and get

$$G_T^f(z^{-1}) = \frac{1.624z^{-1} + 3.638z^{-2} + 1.056z^{-3} + 0.102z^{-4}}{1 + 1.37z^{-1} - 3.195z^{-2} + 1.394z^{-3} + 0.287z^{-4}}. \quad (57)$$

In the simulation, we use an ARX model with 4-th order (while the true model order is 2) to estimate the parameters and get the same model as (57). The frequency responses of $G_T(z^{-1})$ and $G_T^f(z^{-1})$ are plotted in Fig. 3 (only in magnitude). So, in this case, the model estimated in the anti-aliasing filtering procedure is with higher order and not consistent with the true model, but it approximates the true one in frequency domain. But note that this is a noise-free case, so we can say this procedure as not optimal.

### *7.2. Open-loop identification*

**Example 2.** Consider an open-loop system with the second-order true process (53) and the zero-order holding period $T = 1$. The input excitation is $u(m) = \frac{0.1701z^{-1}}{1-0.9149z^{-1}}w(m)$, where $w(m)$ is a zero-mean Gaussian white noise with unit variance.

We construct an output noise which contains a low-pass filtered Gaussian white noise term (contains almost no high frequency noise) and a Gaussian white noise term (contains high frequency noise):

$$v_\Delta(k) = \sigma(H_{lp}(z^{-1})e_1(k) + \alpha e_2(k)) \quad (58)$$

where $e_1(k)$ and $e_2(k)$ are independent Gaussian white noise with unit variance and zero mean. $H_{lp}(z^{-1})$ is a low-pass filter to filter out the high frequency term in the white noise $e_1(k)$,

$$H_{lp}(z^{-1}) = \frac{0.0026z^{-1} + 0.0024z^{-2}}{1 - 1.7997z^{-1} + 0.8007z^{-2}}. \quad (59)$$

We let the sampling period of the output noise to be $T/p$ to simulate continuous-time process, which makes no difference for illustrating the results. The parameter $\sigma$ in (58) is used to adjust the NSR (noise-to-signal ratio, in the simulations the NSR is between 10% to 20% which is normal in real applications). The parameter $\alpha$ is to adjust the ratio of high frequency term in the output noise. By adjusting $\alpha$, we consider two cases:

- $\alpha = 0$, the output noise contains no white noise, thus the high frequency noise is almost zero.
- $\alpha \neq 0$, the output noise contains 30% white noise, thus the high frequency noise is not zero.

**Remark 3.** The case that the output noise contains 30% white noise is not common. The noise structure (58) is, however, common in real applications, where the output noise is usually constructed with a auto-correlated disturbance and a little bit white measurement noise. But in practice, the correlated disturbance can also contain high frequency noise, while here we filter out the high frequency noise in the first term in (58), so we allow the white noise term to be as high as 30%. We use this structure (58) with these two cases to illustrate that the proposed method (or the higher sampling frequency) works on condition that the output has high frequency noise. Nevertheless, we have also done simulations where the output noise is auto-correlated and has high

**Table 1**
Open-loop identification: mean and STD.

| | Case 1: 0% white noise | | Case 2: 30% white noise | |
|---|---|---|---|---|
| True | ASYM | ASYM$_p$ | ASYM | ASYM$_p$ |
| a1: −1.5000 | −1.4946 ± 0.0095 | −1.4947 ± 0.0096 | −1.4917 ± 0.0136 | −1.4956 ± 0.0093 |
| a2: 0.7000 | 0.6947 ± 0.0090 | 0.6944 ± 0.0089 | 0.6928 ± 0.0108 | 0.6954 ± 0.0083 |
| b1: 1.0000 | 1.0009 ± 0.0354 | 1.0021 ± 0.0353 | 0.9937 ± 0.0651 | 1.0000 ± 0.0425 |
| b2: 0.5000 | 0.5165 ± 0.0489 | 0.5144 ± 0.0511 | 0.5351 ± 0.0988 | 0.5144 ± 0.0583 |

**Table 2**
Closed-loop identification.

| | Case 1: 0% white noise | | Case 2: 30% white noise | |
|---|---|---|---|---|
| Ture | ASYM | ASYM$_p$ | ASYM | ASYM$_p$ |
| a1: −1.5000 | −1.4984 ± 0.0065 | −1.4991 ± 0.0061 | −1.4967 ± 0.0133 | −1.4994 ± 0.0071 |
| a2: 0.7000 | 0.6987 ± 0.0052 | 0.6985 ± 0.0048 | 0.6963 ± 0.0136 | 0.6984 ± 0.0063 |
| b1: 1.0000 | 0.9993 ± 0.0035 | 0.9992 ± 0.0032 | 0.9979 ± 0.0104 | 0.9996 ± 0.0049 |
| b2: 0.5000 | 0.5000 ± 0.0025 | 0.5000 ± 0.0015 | 0.5006 ± 0.0101 | 0.5013 ± 0.0028 |

frequency term, the proposed method also performs better than the conventional identification there.

In this example, we applied ASYM$_p$ with $p = 10$. And for comparison, we estimate the plant model $G_T(z^{-1})$ directly from the $T$-sampled data using the original ASYM in Zhu (1998). The results were derived through one hundred runs of Monte-Carlo simulations, the estimated parameters are given in Table 1. One can see that, in Case 1, the two methods perform similar; in Case 2, the proposed method performs better. This implies that the higher sampling frequency can help increase the identification accuracy only when there is high frequency noise.

### 7.3. Closed-loop identification

**Example 3.** Consider again the closed-loop system with the second order true process (53) and the digital controller (54) (the control period is $T = 1$). The reference signal $r(m)$ is a zero-mean Gaussian white noise with unit variance and the NSR is between 10% and 20%.

We apply ASYM$_p$ with $p = 10$ and we also estimate $G_T(z^{-1})$ directly from the $T$-sampled data using ASYM. Again, the two cases in Example 2 are considered. Table 2 contains the parameters estimated from one hundred runs of Monte-Carlo simulations. One can see that, when there is high frequency noise, the proposed method performs even better in closed-loop tests than in open-loop tests, this is consistent with our result in Section 6.

## 8. Conclusion

In this work, we provide analysis for the anti-aliasing filtering procedure in time domain and frequency domain, it is proven that this procedure gives an inconsistent estimate of the desired model. Then we develop an identification method using the over-sampling scheme for both open-loop and closed-loop tests. The proposed method implies anti-aliasing filtering. And moreover, it achieves extra excitation from the high frequency noise in close-loop tests. Also, the two fundamental questions raised in the abstract have been answered.

The result has significant implications in applications. In Example 3, when the output noise contains 30% white noise, the proposed method reduces the standard deviations of the parameters by almost a half. Note that this is achieved without any practical cost, since the high frequency data are there in the DCS/PLC to be used and the computer capacity (memory and speed) has a lot of redundancy. The proposed method can be easily extended to multi-input multi-output (MIMO) plant, as the asymptotic theory can be extended to the MIMO case following the work of Zhu (1989).

However, when a very large $p$ is used, the over-sampling scheme may have numerical difficulties as we have experienced with some prediction error methods, because the model poles will move closer to the unit circle when the sampling frequency increases. This problem can be settled by using continuous-time identification methods to estimate the plant model, which is the direction of our future research.

## Appendix A.

**Proof of Lemma 1.** The proof of (37) follows directly from the proof of Theorem 1 in Sun and Sano (2009).

For the $\Delta$-sampled output noise $v_\Delta(k)$ in (7), define its zero-order held signal $\bar{v}_\Delta(k)$ as

$$\bar{v}_\Delta(mp) = \cdots = \bar{v}_\Delta((m+1)p-1) = v_\Delta(mp) = v(m) \tag{A.1}$$

then we have

$$u_\Delta(k) = \frac{-K(z^{-p})}{1+K(z^{-p})G_T(z^{-p})}\bar{v}_\Delta(k) \tag{A.2}$$

and

$$S_{u_\Delta,e_\Delta}(\alpha_l,\omega) = \frac{-K(e^{ip\omega})}{1+K(e^{ip\omega})G_T(e^{ip\omega})}S_{\bar{v}_\Delta,e_\Delta}(\alpha_l,\omega). \tag{A.3}$$

One the other hand,

$$\begin{aligned} &S_{\bar{v}_\Delta,e_\Delta}(\alpha_l,\omega) \\ &= \sum_{\tau=-\infty}^{\infty}\left(\frac{1}{p}\sum_{k=0}^{p-1}R_{\bar{v}_\Delta,e_\Delta}(k+mp,\tau)e^{-i\alpha_l k}\right)e^{-i\omega\tau} \\ &= \frac{1}{p}\sum_{\tau=-\infty}^{\infty}(E\left\{\bar{v}_\Delta(mp+\tau)e_\Delta(mp)\right\}+\cdots \\ &+E\left\{\bar{v}_\Delta(mp+p-1+\tau)e_\Delta(mp+p-1)\right\}e^{-i\alpha_l(p-1)})e^{-i\omega\tau} \end{aligned} \tag{A.4}$$

in the last term of (A.4), let $\tau = \mu p + j$, $\tau = \mu p - 1 + j$, $\cdots$, $\tau = \mu p - (p-1) + j$ sequentially and $j = 0, \cdots, p-1$, then (A.4) becomes

$$\frac{1}{p}\sum_{j=0}^{p-1} e^{-ij\omega}\Big( \sum_{\mu=-\infty}^{\infty} E\left\{\bar{v}_{\Delta}(mp+\mu p)e_{\Delta}(mp)\right\} e^{-i(\omega_l-\alpha)\mu p}$$
$$+ \sum_{\mu=-\infty}^{\infty} E\left\{\bar{v}_{\Delta}(mp+\mu p)e_{\Delta}(mp+1)\right\} e^{-i(\omega-\alpha_l)(\mu p-1)}$$
$$+\cdots+ \sum_{\mu=-\infty}^{\infty} (E\left\{\bar{v}_{\Delta}(mp+\mu p)e_{\Delta}(mp+p-1)\right\} \cdot e^{-i(\omega_l-\alpha)(\mu-(p-1))}\Big))$$

finally let $\mu p = -\tau$, $\mu p - 1 = -\tau$, $\cdots$, $\mu p - (p-1) = -\tau$ sequentially, then (A.4) becomes

$$S_{\bar{v}_{\Delta},e_{\Delta}}(\alpha_l,\omega) = \frac{1}{p}\sum_{j=0}^{p-1} e^{-ij\omega} \cdot \sum_{\tau=-\infty}^{\infty} E\left\{e_{\Delta}(mp+\tau)\bar{v}_{\Delta}(mp)\right\} e^{i(\omega-\alpha_l)\tau}$$
$$= \frac{1}{p}\sum_{j=0}^{p-1} e^{-ij\omega} \sum_{\tau=-\infty}^{\infty} (E\left\{e_{\Delta}(k+\tau)\bar{v}_{\Delta}(k)\right\} e^{i(\omega-\alpha_l)\tau}$$
$$= \frac{\lambda_{\Delta}}{p}\sum_{j=0}^{p-1} e^{-ij\omega} \cdot H_{\Delta}(e^{-i(\omega-\alpha_l)}) \qquad \text{(A.5)}$$

combine (A.5) and (A.3), then (39) is derived. □

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